University of Wisconsin-Madison
Math 340 - Fall 2010
Linear Algebra

Final Exam

Exercise 1:

(1) Let $L_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map satisfying $\dim\mathrm{Ker}L_1 = 3$. Find L_1.
(2) Let $L_2 : \mathbb{R}_3[X] \to \mathbb{R}_2[X]$ be a linear map satisfying $\dim\mathrm{Ker}L_2 = 1$. Find its image $\mathrm{Im}L_2$.
(3) Is there any linear map $L_3 : \mathbb{R}^3 \to \mathbb{R}_3[X]$ such that $\mathrm{Im}L_3 = \mathbb{R}_3[X]$?

Exercise 2: Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L((-1,1)) = (1,-3)$ and $L((-1,3)) = (0,1)$. Find $L((1,1)$.
Exercise 3: Prove or disprove that the following maps are linear.

(1) $L_1 : \mathbb{R}_4[X] \to \mathbb{R}$ with $L_1(P) = 0$.
(2) $L_2 : \mathbb{R}^3 \to \mathbb{R}^3$, satisfying $L((2,-1,1)) = (1,0,1)$ and $L((-2,1,-1)) = (0,1,4)$.
(3) $L_3 : \mathbb{R}^3 \to \mathbb{R}^3$, with $L_3((x,y,z)) = (x, xy, y+2z)$.

Exercise 4: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map defined by

$$L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} y \\ 0 \\ x+y \end{pmatrix}$$

Let $\mathcal{C}$ be the canonical basis and consider $\mathcal{B} = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ with,

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(1) Check that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.
(2) Find the matrix of L relative to the canonical basis $\mathcal{C}$, $[L]_{\mathcal{C}}^{\mathcal{C}}$.
(3) Find the transition matrix, $[I]_{\mathcal{B}}^{\mathcal{C}}$, for changing from $\mathcal{B}$ to $\mathcal{C}$.

(4) Find the transition matrix, $[I]_{\mathcal{C}}^{\mathcal{B}}$, for changing from $\mathcal{C}$ to $\mathcal{B}$.
(5) Find the matrix of L relative to the basis $\mathcal{B}$, $[L]_{\mathcal{B}}^{\mathcal{B}}$.

Exercise 5: Let $M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \mathbb{R} \right\}$ the space of 2×2 matrices. Let $L : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the transformation defined by $L(A) = A^t$.

(1) Show that L is linear.
(2) Show that $\mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis of $M_{2\times 2}(\mathbb{R})$.
(3) Find the matrix $[L]_{\mathcal{C}}^{\mathcal{C}}$ of L relative to $\mathcal{C}$.

Exercise 6: Consider the space $W = span\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix} \right\}$.

(1) Show that $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$ is an orthonormal basis of W.

(2) Compute $\mathrm{proj}_W \left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$.

Exercise 7:

(1) Let $L_1 : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$ be the linear map defined by $L(P) = P'$.
   (a) Compute $L_1 \circ L_1(a + bX + cX^2)$.
   (b) Compute $L_1 \circ L_1 \circ L_1(a + bX + cX^2)$.

(2) Let $L_2 : \mathbb{R}_2[X] \to \mathbb{R}_2[X]$ be the linear map defined by $L_2(a + bX + cX^2) = c + aX + bX^2$.
   (a) Compute $L_2 \circ L_2(a + bX + cX^2)$.
   (b) Compute $L_2 \circ L_2 \circ L_2(a + bX + cX^2)$.
   (c) Compute $\underbrace{L_2 \circ L_2 \circ \cdots \circ L_2}_{n \text{ times}}(a + bX + cX^2)$ for any n.

Exercise 8: Consider the following inner product on $\mathbb{R}_2[X]$:

$$< P, Q >:= \int_{-1}^{1} P(t)Q(t)dt.$$

(1) Find a basis of $\{1\}^\perp := \{P \in \mathbb{R}_2[X], < P, 1 >= 0\}$.

(2) What is the dimension of $\{1\}^\perp$?